1. Freddy colors a 2×2 grid of squares using red, yellow, green, and blue, with each square colored solid. He does not want to use the same color for adjacent squares. How many ways can Freddy color the grid?

   Answer: $\boxed{\mathbf{84}}$

   Solution: There are 4 ways to color the upper-left square. To color the remaining squares, we have two cases:

   Case 1: the upper-right and the lower-left square are the same color.

   There are 3 ways to color the upper-right and lower-left square (every color except the color of the upper-left square), and 3 ways to color the lower-right square (every color except the color of the upper-right and lower-left square). Thus, for this case, there are $4 \cdot 3 \cdot 3 = 36$ ways to color the grid.

   Case 2: the upper-right and the lower-left square are different colors.

   There are 3 ways to color for the upper-right square (every color except the color of the upper-left square), 2 ways to color the lower-left square (every color except the color of the upper-left and upper-right square), and 2 ways to color the lower-right square (every color except the color of the upper-right square and the color of the lower-left square). Thus, for this case, there are $4 \cdot 3 \cdot 2 \cdot 2 = 48$ ways to color the grid.

   So, in total, there are $36 + 48 = \boxed{\mathbf{84}}$ ways to color the grid.

2. Fazbear is writing his dreaded dissertation on the prevalence of bears in German and Polish heraldry. If he writes at a pace of 12 words per minute for half an hour and then at a pace of 17 words per minute for the next twenty minutes, what was his average writing speed, in words per minute?

   Answer: $\boxed{\mathbf{14}}$

   Solution: Fazbear's average writing speed is

   $$\frac{12 \text{ words per minute} \cdot 30 \text{ minutes} + 17 \text{ words per minute} \cdot 20 \text{ minutes}}{50 \text{ minutes}} = \boxed{\textbf{14 words per minute}}.$$

3. Eric is selling refrigerators to afford his Nueva tuition. He can pay off his debt alone by selling for 12 hours. After he has been selling for 5 hours, he calls his friend Jason, who starts selling golden apples at his own house. Working together, Eric's tuition is fully paid off after another 3 hours. If Jason were to pay off his own tuition, which is the same as Eric's, how long would he have to sell his apples for alone?

   Answer: $\boxed{\mathbf{9}}$

   Solution: Eric can pay off $\frac{1}{12}$ of his tuition in an hour. When he gets help from Jason, he can pay off the remaining $\frac{12-5}{12} = \frac{7}{12}$ of his tuition in 3 hours. Together, they can pay off $\frac{\frac{7}{12}}{3} = \frac{7}{36}$ of his tuition in an hour.

   To find Jason's paying rate, we subtract Eric's paying rate from Jason and Eric's combined paying rate. Thus, Jason can pay off $\frac{7}{36} - \frac{1}{12} = \frac{1}{9}$ of his tuition in hour, and so it would take Jason $\boxed{\mathbf{9}}$ hours to pay off all of his tuition.

4. There is a magical coin that never lands on heads more than three times in a row and never lands on tails more than two times in a row. This coin can only be flipped ten times before it will disappear forever. How many different flip sequences of heads and tails can be attained?

   Answer: $\boxed{\mathbf{328}}$

   Solution: Let an n-sequence be a sequences of n flips such that the coin never lands on heads more than three times in a row and never lands on tails more than two times in a row. Let $F_H(n)$ be the number of n-sequences that end with a head, and let $F_T(n)$ be the number of n-sequences that end with a tail. We want to find $F_H(10) + F_T(10)$.

   We can find quickly that:

   $$F_H(1) = 1, F_T(1) = 1$$

$$F_H(2) = 2, F_T(2) = 2$$
$$F_H(3) = 4, F_T(3) = 3$$
$$F_H(4) = 6, F_T(4) = 6$$

We now find $F_H(n)$ and $F_T(n)$ for $n > 4$.

First, we find $F_H(n)$. If we have an $(n-1)$-sequence that ends in a tail, we can add a head to get an n-sequence. If we have an $(n-1)$-sequence that ends in a head, we can add a head to get either an n-sequence or a sequence of n flips that ends in 4 heads. The number of possible sequences of the latter is $F_T(n-4)$, as it is equivalent to adding 4 heads to an $(n-4)$-sequence that ends in a tail. Thus, we define $F_H(n)$ as:

$$F_H(n) = F_H(n-1) + F_T(n-1) - F_T(n-4)$$

We then find $F_T(n)$. If we have an $(n-1)$-sequence that ends in a head, we can add a tail to get an n-sequence. If we have an $(n-1)$-sequence that ends in a tail, we can add a tail to get either an n-sequence or a sequence of n flips that ends in 3 tails. The number of possible sequences of the latter is $F_H(n-3)$, as it is equivalent to adding 4 tails to an $(n-4)$-sequence that ends in a head. Thus, we define $F_H(n)$ as:

$$F_T(n) = F_H(n-1) + F_T(n-1) - F_H(n-3)$$

We can now find $F_H(10) + F_T(10)$. We have:

$$F_H(5) = 11, F_T(5) = 10$$
$$F_H(6) = 19, F_T(6) = 17$$
$$F_H(7) = 33, F_T(7) = 30$$
$$F_H(8) = 57, F_T(8) = 52$$
$$F_H(9) = 99, F_T(9) = 90$$
$$F_H(10) = 172, F_T(10) = 156$$

We get that $F_H(10) + F_T(10) = \boxed{\mathbf{328}}$.

5. In triangle $\triangle ABC$, let D, E, and F be the feet of the altitude from vertices A, B, and C to sides BC, AC, and AB, respectively. If $AD = 10$, $BE = 9$, and $CD = 8$, find the area of $\triangle ABC$.

   Answer: $\boxed{\mathbf{9\sqrt{41}}}$

   Solution: Observing that $\angle ADC = 90^\circ$, we have $AC = \sqrt{AD^2 + CD^2} = \sqrt{10^2 + 8^2} = 2\sqrt{41}$. The altitude of $\triangle ABC$ to side AC is $BE = 9$, meaning that the area of $\triangle ABC$ is

$$[\triangle ABC] = \frac{AC \cdot BE}{2} = \frac{2\sqrt{41} \cdot 9}{2} = \boxed{\mathbf{9\sqrt{41}}}.$$

6. Three children want to distribute 12 indistinguishable pieces of candy. Alice wants at least 1 piece of candy, Bob wants at least 2 pieces of candy, and Carol wants at least 3 pieces of candy. How many ways can the pieces of candy be distributed such that each child gets their desired amount of candy?

   Answer: $\boxed{\mathbf{28}}$

   Solution: First, we give the three children the minimum amount of candy that they want. After doing this, we have left $12 - 1 - 2 - 3 = 6$ pieces of candy. We now need to find how many ways to distribute the remaining 6 pieces of candy.

   Using the "stars and bars" method, we can rephrase this problem. We imagine putting the six candies in a line, placing 2 indistinguishable dividers between the pieces of candy, splitting the candies into 3 sub-collections, and giving each child a sub-collection. There are 7 ways to place the first divider, and 8 ways to place the second divider. There are then $7 \cdot 8$ ways to place the two dividers, but since the dividers are indistinguishable, we divide this number by 2!. Thus, the total number of ways to place the dividers is $\frac{7 \cdot 8}{2!} = \boxed{\mathbf{28}}$.

7. An infinite sequence $a_1, a_2, a_3, \ldots$ is defined recursively so that $a_1 = 2 - \sqrt{3}$, $a_2 = 2 + \sqrt{3}$, and

$$a_{n+2} = \frac{2(a_n + a_{n+1})(1 - a_n a_{n+1})}{1 + (a_n a_{n+1})^2 - (a_n + a_{n+1})^2}$$

for all positive integers n. Compute the value of $(|a_{2023}| + |a_{2024}| + |a_{2025}|)^2$.

Answer: $\boxed{\mathbf{12}}$

Solution: We can verify that $a_1 = \tan \frac{\pi}{12}$, $a_2 = \tan \frac{5\pi}{12}$, and $a_3 = 0$. Additionally, observe that

$$\tan(2\alpha + 2\beta) = \frac{2(\tan\alpha + \tan\beta)(1 - \tan\alpha\tan\beta)}{1 + (\tan\alpha\tan\beta)^2 - (\tan\alpha + \tan\beta)^2 - 2\tan\alpha\tan\beta}.$$

Since $a_3 = 0$, it thus follows that $a_4 = \tan \frac{5\pi}{6}$, $a_5 = \tan \frac{5\pi}{3} = \tan \frac{2\pi}{3}$, and $a_6 = 0$.

Repeating this procedure, we get $a_{3n+7} = \tan \frac{4\pi}{3} = \tan \frac{\pi}{3} = \sqrt{3}$, $a_{3n+8} = \tan \frac{2\pi}{3} = -\sqrt{3}$, and $a_{3n+9} = 0$ for all nonnegative integers n. In other words, the subsequence $a_7, a_8, a_9, \ldots$ cycle among $\sqrt{3}$, $-\sqrt{3}$, and 0 in that order. Hence $(a_{2023}, a_{2024}, a_{2025}) = (\sqrt{3}, -\sqrt{3}, 0)$, implying that $(|a_{2023}| + |a_{2024}| + |a_{2025}|)^2 = \boxed{\mathbf{12}}$.

8. Daniel, Ethan, and Felix have lunch at Karl's Kitchen and agree to split the \$100 bill. If Daniel insists on paying either \$7, \$11, \$13, or \$17, Ethan insists on paying in \$8 bills, and Felix insists on paying an amount between \$37 and \$73, in how many ways can the friends split the bill?

Answer: $\boxed{\mathbf{19}}$

Solution: We approach this problem using generating functions. Daniel will only pay \$7, \$11, \$13, or \$17, so his generating function is

$$x^7 + x^{11} + x^{13} + x^{17} = x^7(1 + x^4)(1 + x^6).$$

Ethan will only pay in \$8 bills, so his generating function is

$$1 + x^8 + x^{16} + \cdots = \frac{1}{1 - x^8}.$$

Felix will only pay an amount between \$37 and \$73 so his generating function is

$$x^{37} + x^{38} + x^{39} + \cdots + x^{73} = x^{37}(1 + x + x^2 + \cdots + x^{36}) = \frac{x^{37}(1 - x^{37})}{1 - x}.$$

It suffices to find the coefficient of x^{100} in the combined generating function given by $\frac{x^{44}(1+x^4)(1+x^6)(1-x^{37})}{(1-x^8)(1-x)}$, which is equivalent to finding the coefficient of x^{56} in the expansion of

$$\frac{(1 + x^4)(1 + x^6)(1 - x^{37})}{(1 - x^8)(1 - x)} = (1 + x^4 + x^6 + x^{10} - x^{37} - x^{41} - x^{43} - x^{47})(1 + x^8 + x^{16} + \cdots)(1 + x + x^2 + \cdots).$$

One can check that the coefficient of x^n in the expansion of $(1 + x^8 + x^{16} + \cdots)(1 + x + x^2 + \cdots)$ is simply $\lfloor \frac{n}{8} \rfloor + 1$, meaning that the coefficient of x^{56} in the expansion of $x^n(1 + x^8 + x^{16} + \cdots)(1 + x + x^2 + \cdots)$ is given by the function

$$f(n) = \left\lfloor \frac{56 - n}{8} \right\rfloor + 1 = \left\lfloor -\frac{n}{8} \right\rfloor + 8.$$

The coefficient of x^{56} in $P(x)$ is thus

$$f(0) + f(4) + f(6) + f(10) - f(37) - f(41) - f(43) - f(47) = \boxed{\mathbf{19}}.$$

9. Let p, q, and r be the three positive roots of the polynomial $x^3 - 10x^2 + 31x - 29$. What is the area of the triangle with side lengths p, q, and r, given that it exists?

Answer: $\boxed{\sqrt{5}}$

Solution: Using Heron's formula, the area of this triangle can be expressed as $\sqrt{s(s-p)(s-q)(s-r)}$, where $s = \frac{p+q+r}{2}$. Since $p+q+r = 10$ by Vieta's Formulas, it follows that $s = 5$.

Additionally, since $(s-a)(s-b)(s-c)$ has the same roots as $x^3 - 10x^2 + 31x - 29$, we have that

$$(s-a)(s-b)(s-c) = s^3 - 10s^2 + 31s - 29 = 5^3 - 10 \cdot 5^2 + 31 \cdot 5 - 29 = 1.$$

The area of our desired triangle is therefore

$$\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{5 \cdot 1} = \boxed{\sqrt{5}}.$$

10. There are two breadboards on the table, with one containing 99 red LEDs, numbered from 2 to 100, and one containing 99 blue LEDs, likewise numbered from 2 to 100. Initially, all of the LEDs are turned off. When a button is pressed, the following algorithm is executed:

- A robot writes down the minimal integer k between 2 and 100, inclusive, such that the blue LED numbered k is off.
- Each red LED whose number is a multiple of k is toggled: LEDs that are currently off are turned on, and LEDs that are currently on are turned off.
- Each blue LED whose number is a multiple of k and is currently off is turned on.

Let m be the minimal number of presses of the button for all blue LEDs to turn on, and let n be the number of red LEDs that are turned on when this occurs. Compute $100m + n$.

Answer: $\boxed{2543}$

Solution. First we consider the blue LEDs. Observe that the algorithm used to turn on blue LEDs implies that the value of k at the ith button press is simply the ith prime number. Since there are 25 prime numbers between 2 and 100, inclusive, it follows that $m = 25$.

We note that a red LED corresponding to j will be toggled when the button is pressed only if k is a prime factor of j. Hence, the red LED corresponding to j will lit after the 25th button press only if i has an odd number of distinct prime factors. The greatest number of prime factors j can have is 3, as $2 \cdot 3 \cdot 5 < 100 < 2 \cdot 3 \cdot 5 \cdot 7$, so we must find when j has 1 or 3 distinct prime factors.

First, we consider the case where j has exactly 1 distinct prime factor, so that $j = p^a$ for some prime p and positive integer a. If $a = 1$, then p has 25 options, as there are 25 primes under 100. If $a = 2$, then $p \leq 10$, so it has 4 options. If $a \in \{3, 4\}$, then p has 2 options. If $a \in \{5, 6\}$, then p has 1 option. If $a \geq 7$, then $2^7 > 100$ implies that no p works. This yields $25 + 4 + 2 \cdot 2 + 2 \cdot 1 = 35$ cases here.

Now, consider the case where j has exactly 3 distinct prime factors, so that $j = p^a q^b r^c$, where p, q, and r are primes and a, b, and c are positive integers. Suppose without losing generality that $p < q < r$, we find that having $p \geq 3$ yields $j \geq 3 \cdot 5 \cdot 7 > 100$, implying that $p = 2$. Similarly, if $q \geq 7$, then $j \geq 2 \cdot 7 \cdot 11 > 154$, so that $q \in \{3, 5\}$. We consider each case separately.

- If $(p, q) = (2, 3)$, then $6 \mid j$. Then one can check $r \in \{5, 7, 11, 13\}$. Then the values of j under these options are $2^1 3^1 5^1 = 30$, $2^2 3^1 5^1 = 60$, $2^1 3^2 5^1 = 90$, $2^1 3^1 7^1 = 42$, $2^2 3^1 7^1 = 84$, $2^1 3^1 11^1 = 66$, and $2^1 3^1 13^1 = 78$. This gives 7 options.
- If $(p, q) = (2, 5)$, so that $10 \mid j$, then we must have $r = 7$ and $j = 2^1 5^1 7^1 = 70$. Hence there is 1 option for this case.

There are therefore totally $7 + 1 = 8$ possibilities when j has 3 distinct prime factors. In total, it follows that there will be $35 + 8 = 43$ lit red LEDs when all blue LEDs are finally lit, so that $n = 43$.

Our answer is therefore $100m + n = \boxed{2543}$.