1. Let ABC be a triangle where $\overline{AB} = 7$ and $\overline{AC} = 13$. Let D, E, and F all be on $\overline{BC}$ such that $\overline{AD}$ is an altitude from A, $\overline{AE}$ is an angle bisector of $\angle BAC$, and $\overline{AF}$ is a median from A. If $\overline{DE} = \overline{EF}$, find $\overline{BC}^2$

   Answer: $\boxed{\mathbf{200}}$

   Solution:

   Using the angle bisector theorem, we have that $\frac{\overline{BE}}{\overline{CE}} = \frac{\overline{AB}}{\overline{AC}} = \frac{7}{13}$. So, we can let $\overline{BE} = 7x$ and $\overline{CE} = 13x$.

   Since $\overline{AF}$ is a median, we have that $\overline{BE} = 10x$ and $\overline{CE} = 10x$. Since $\overline{DE} = \overline{EF}$, we also have that $\overline{BD} = 4x$ and $\overline{CD} = 16x$.

   Using the Pythagorean theorem, we get:

$$\begin{aligned} \overline{AB}^2 - \overline{BD}^2 &= \overline{AC}^2 - \overline{BC}^2 \\ 49 - 16x^2 &= 169 - 256x^2 \\ 240x^2 &= 120 \\ x^2 &= \frac{1}{2} \end{aligned}$$

   Note that $\overline{BC} = 20x$. So, the answer to this problem is $\overline{BC}^2 = 400x^2 = \boxed{\mathbf{200}}$.

2. Suppose Bill has jars of 4, 7, and 10 jelly beans. Assuming that he has enough jars, what is the largest number of jelly beans that Bill cannot make using these jars?

   Answer: $\boxed{\mathbf{13}}$

   Solution:

   Using the Chicken McNugget Theorem, we have that any number no less than $(2-1)(5-1) = 6$ can be expressed in the form $2a + 5b$, where a and b are nonnegative integers. So, all even numbers no less than 12 can be expressed in the form $4a + 10b$, meaning that all even numbers of jelly beans no less than 12 can be made from the jelly bean jars.

   This also means that all odd numbers of jelly beans no less than $12 + 7 = 19$ can be made from the jelly bean jars. We now check all odd numbers less than 19 to find greatest number of jelly beans which cannot be made from the jars.

   We find that $17 = 7 + 10$ and $15 = 4 + 4 + 7$ can be expressed as $4a + 7b + 10c$, whereas 13 cannot. So, the largest number of jelly beans that Bill cannot make using these jars is $\boxed{\mathbf{13}}$.

3. For all subsets of $\{1, ..., 100\}$, the expected number of consecutive values in the set (for example, $\{2, 3, 4\}$ has 2 consecutive values) is $\frac{a}{b}$, where a and b are relatively prime. Find $a + b$.

   Answer: $\boxed{\mathbf{103}}$

   Solution:

   Let N be the number of consecutive values in S, where S is a subset of $\{1, ..., 100\}$. For all $1 \leq i \leq 99$, let $N_i = 1$ if $i, i + 1 \in S$, and $N_i = 0$ if otherwise. Note that $E[N] = \sum_{i=1}^{99} E[N_i]$.

   For any $1 \leq i \leq 99$, we have that $E[N_i] = \frac{2^{98}}{2^{100}} = \frac{1}{4}$, since there are 2^{98} subsets out of the 2^{100} subsets of

$\{1, ..., 100\}$ which contain both i and $i+1$. Thus:

$$\begin{aligned} E[N] &= \sum_{i=1}^{99} E[N_i] \\ &= \sum_{i=1}^{99} \frac{1}{4} \\ &= \frac{99}{4} \end{aligned}$$

So, the answer to this problem is $\boxed{\mathbf{103}}$.

4. The solution of the equation $4^{x+8} = 9^x$ can be expressed in the form $\log_a(4^4)$. What is a? Express your answer in terms of a fraction in simplest form

   Answer: $\boxed{\frac{3}{2}}$

   Solution:

   By taking the natural log of both sides of $4^{x+8} = 9^x$, we have:

$$\begin{aligned} \log_2 4^{x+8} &= \log_2 9^x \\ 2(x+8) &= x \log_2 9 \\ (\log_2 9 - 2)x &= 16 \\ x &= \frac{16}{\log_2 9 - 2} \\ x &= \frac{\log_2 2^{16}}{\log_2 \frac{9}{4}} \\ x &= \log_{\frac{9}{4}}(2^{16}) \\ x &= \log_{\frac{3}{2}}(4^4) \end{aligned}$$

   So, the answer to this problem is $\boxed{\frac{3}{2}}$.

5. Given three positive numbers a, b, c, the minimum possible value of $\frac{a^2}{(b+c)^2+a^2} + \frac{b^2}{(c+a)^2+b^2} + \frac{c^2}{(a+b)^2+c^2}$ is $\frac{x}{y}$, where x and y are relatively prime positive integers. Find $x+y$.

   Answer: $\boxed{\mathbf{8}}$

   Solution:

   Let $S = \frac{a^2}{(b+c)^2+a^2} + \frac{b^2}{(c+a)^2+b^2} + \frac{c^2}{(a+b)^2+c^2}$. We can express S as $\frac{1}{1+(\frac{b+c}{a})^2} + \frac{1}{1+(\frac{a+c}{b})^2} + \frac{1}{1+(\frac{a+b}{c})^2}$.

   Let $x = \frac{b+c}{a}$, $y = \frac{a+c}{b}$, $z = \frac{a+b}{c}$. So, $S = \frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{z^2}$. Using the AM-HM inequality, we have:

$$\begin{aligned} \frac{1+x^2+1+y^2+1+z^2}{3} &\geq \frac{3}{\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2}} \\ &= \frac{3}{S} \end{aligned}$$

In order to minimize S, we'd like to have $\frac{1+x^2+1+y^2+1+z^2}{3} = \frac{3}{S}$, which happens when $1+x^2 = 1+y^2 = 1+z^2$. Let $x = y = z = k$. Then, we have that

$$\begin{aligned} b + c &= ak \\ a + c &= bk \\ a + b &= ck \end{aligned}$$

Adding up these equations together gives us $2(a+b+c) = k(a+b+c)$, thus $k = 2$. Thus, can calculate S.

$$\begin{aligned} S &= \frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2} \\ &= \frac{1}{1+2^2} + \frac{1}{1+2^2} + \frac{1}{1+2^2} \\ &= \frac{3}{5} \end{aligned}$$

So, the answer is $\boxed{\mathbf{8}}$

6. Let $\triangle ABC$ be a triangle with $\overline{AB} = 26$, $\overline{AC} = 17$, and $\overline{BC} = 25$. Define points $H_1, H_2, \ldots$ as follows: Let H_1 be the altitude from B to $\overline{AC}$ (the point on $\overline{AC}$ that angle $\angle AH_1B$ is right), let H_2 be the altitude from H_1 to $\overline{AB}$, and let H_{n+2} be the altitude from H_{n+1} to $\overline{AH_n}$ for $n \geq 2$. Find the infinite sum $\overline{BH_1} + \overline{H_1H_2} + \overline{H_2H_3} + \ldots$

   Answer: $\boxed{\mathbf{39}}$

   Solution:

   We have that the area of the triangle $\triangle ABC$ is $\sqrt{34 \cdot 8 \cdot 17 \cdot 9} = 204$. So, we have that $\overline{BH_1} = \frac{2 \cdot 204}{17} = 24$.

   To find $\overline{H_1H_2}$, note that $\frac{\overline{H_1H_2}}{\overline{BH_1}} = \frac{\overline{AH_1}}{\overline{AB}}$. We find that $\overline{AH_1} = \sqrt{\overline{AB}^2 - \overline{BH_1}^2} = \sqrt{26^2 - 24^2} = 10$. So, we have that $\overline{H_1H_2} = \frac{\overline{AH_1} \cdot \overline{BH_1}}{\overline{AB}} = \frac{10 \cdot 24}{26} = \frac{120}{13}$.

   Note that $\overline{BH_1}, \overline{H_1H_2}, \overline{H_2H_3}, \ldots$ is a geometric sequence with common ratio $\frac{\frac{120}{13}}{24} = \frac{5}{13}$ So, the infinite sum $\overline{BH_1} + \overline{H_1H_2} + \overline{H_2H_3} + \ldots$ is:

$$\begin{aligned} \overline{BH_1} + \overline{H_1H_2} + \overline{H_2H_3} + \ldots &= 24 + 24\left(\frac{5}{13}\right) + 24\left(\frac{5}{13}\right)^2 + \ldots \\ &= \frac{24}{1 - \frac{5}{13}} \\ &= \boxed{\mathbf{39}} \end{aligned}$$

7. Let $a_0, a_1, a_2, \ldots$ be an arithmetic sequence, and let $g_0, g_1, g_2, \ldots$ be a geometric sequence. Given that $a_0 = 108$, $g_0 = 2$, $a_{10} = g_6$, and $a_2 = 7g_3$, find the sum of all possible values of $a_1 + g_1$.

   Answer: $\boxed{\mathbf{363}}$

   Solution:

   Let the common difference in the arithmetic sequence be d, and the common ration in the geometric sequence be r. So, we have:

$$a_0 + 10d = g_0 r^6 \tag{1}$$
$$a_0 + 2d = 7g_0 r^3 \tag{2}$$

From equation (2), we get $d = \frac{7g_0r^3 - a_0}{2}$. We substitute this equation in equation (1) to get:

$$\begin{aligned} a_0 + 10\left(\frac{7g_0r^3 - a_0}{2}\right) &= g_0r^6 \\ a_0 + 35g_0r^3 - 5a_0 &= g_0r^6 \\ g_0r^6 - 35g_0r^3 + 4a_0 &= 0 \end{aligned}$$

Using the quadratic equation, we have:

$$\begin{aligned} r^3 &= \frac{35g_0 \pm \sqrt{(35g_0)^2 - 4(g_0)(4a_0)}}{2g_0} \\ &= \frac{70 \pm \sqrt{(70)^2 - 4(2)(4 * 108)}}{4} \\ &= \frac{70 \pm \sqrt{4900 - 3456}}{4} \\ &= \frac{70 \pm \sqrt{1444}}{4} \\ &= \frac{70 \pm 38}{4} \\ &= 8, 27 \end{aligned}$$

We substitute the first value of r^3 into equation (2):

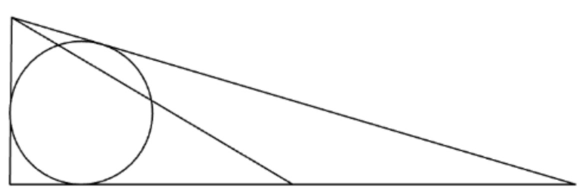
$$\begin{aligned} 108 + 2d &= 7(2)(8) \\ 108 + 2d &= 112 \\ d &= 2 \end{aligned}$$

So, one possible value of (a_1, g_1) is $(110, 4)$. We then substitute the second value of r^3 into equation (2):

$$\begin{aligned} 108 + 2d &= 7(2)(27) \\ 108 + 2d &= 378 \\ d &= 135 \end{aligned}$$

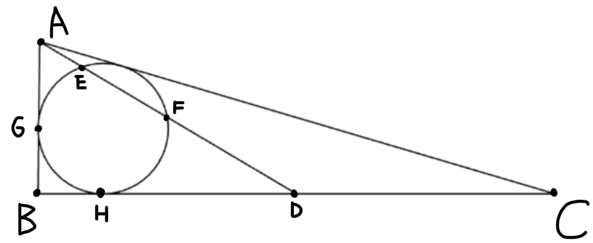
So, another possible value of $(243, 6)$. So, the answer is $110 + 4 + 243 + 6 = \boxed{\mathbf{363}}$.

8. $\triangle ABC$ is a triangle with longest side $\overline{AC}$ and $BC = 24$, and point D lies on BC such that $\overline{BD} = \overline{CD}$. $\overline{AD}$ intersects the incircle of $\triangle ABC$ at 2 points: E and F, with E closer to A than F, and the ratio of $\overline{AE} : \overline{EF} : \overline{FD}$ is $1 : 2 : 3$. If the radius of the incircle of $\triangle ABC$ can be written as $\frac{a\sqrt{b}}{c}$ where a, b, and c are positive integers, b is not divisible by any perfect square greater than 1, and a and c are relatively prime, find $a + b + c$.



Answer: $\boxed{\mathbf{22}}$

Solution:



Let $\overline{AE} = x$. Then, $\overline{EF} = 2x$ and $\overline{FD} = 3x$. Using Power of a Point Theorem, we find that $\overline{AG} = \sqrt{3}x$ and $\overline{DH} = \sqrt{15}x$. We also have that $\overline{GB} = \overline{BH} = 12 - \sqrt{15}x$ and $\overline{AC} = \overline{AG} + \overline{CH} = (\sqrt{3} + \sqrt{15})x + 12$.

Using Stewart's Theorem, we have:

$$\begin{aligned}
\overline{AC}^2 \cdot \overline{DB} + \overline{AB}^2 \cdot \overline{DC} &= \overline{BC}(\overline{AD}^2 + \overline{DB} \cdot \overline{DC}) \\
12(((\sqrt{3} + \sqrt{15})x + 12)^2 + ((\sqrt{3} - \sqrt{15})x + 12)^2) &= 24(36x^2 + 144) \\
(((\sqrt{3} + \sqrt{15})x + 12)^2 + ((\sqrt{3} - \sqrt{15})x + 12)^2) &= 72x^2 + 288 \\
36x^2 + 48\sqrt{3}x + 288 &= 72x^2 + 288 \\
48\sqrt{3} &= 36x \\
x &= \frac{4\sqrt{3}}{3}
\end{aligned}$$

We now find the sides of $\triangle ABC$. We have that:

$$\begin{aligned}
\overline{AB} &= (\sqrt{3} - \sqrt{15})x + 12 \\
&= (\sqrt{3} - \sqrt{15})\left(\frac{4\sqrt{3}}{3}\right) + 12 \\
&= 16 - 4\sqrt{5} \\
\overline{AC} &= (\sqrt{3} + \sqrt{15})x + 12 \\
&= (\sqrt{3} + \sqrt{15})\left(\frac{4\sqrt{3}}{3}\right) + 12 \\
&= 16 + 4\sqrt{5}
\end{aligned}$$

The semiperimeter of the triangle is $\frac{56}{2} = 28$. So, the inradius is:

$$\begin{aligned}
r &= \frac{[\triangle ABC]}{28} \\
&= \frac{\sqrt{28(28 - (16 - 4\sqrt{5}))(28 - (16 + 4\sqrt{5}))(28 - 24)}}{28} \\
&= \frac{32\sqrt{7}}{28} \\
&= \frac{8\sqrt{7}}{7}
\end{aligned}$$

Thus, the answer is $8 + 7 + 7 = \boxed{\mathbf{22}}$.

9. Find the amount of nonempty subsets of $(1, 2, 3, \ldots, 20)$ such that any two (not necessarily distinct) elements differ by a factor of at most 2, and no two elements are consecutive. For example, the subsets $(3, 5)$, $(5, 8, 10)$, are valid subsets, while the subsets $(1, 3)$, $(3, 9, 11, 16, 19)$, and $(6, 7, 11)$ are not valid.

Answer: $\boxed{\mathbf{374}}$

Solution:

Suppose that we want to find a subset which satisfies these conditions where the minimum and maximum value differ by n. Let $f(n)$ be the number of such possible subsets.

To calculate $f(n)$, we first set a to be the minimum value and $a+n$ to be the maximum value of our subset. We then figure out the number of ways to have $0, 1, 2, 3, \ldots$ numbers in between a and $a+n$ such that there are no consecutive numbers. We add these numbers up and multiply by the number of possible values of a such that $a+n \leq 2a$ and $a+n \leq 20$.

Note that $n \neq 1$, otherwise there will be consecutive values. For $n = 0$, we have $f(n) = 20$. For even $n \geq 2$, we have:

$$f(n) = (21-2n)\left(1 + \binom{n-3}{1} + \binom{n-4}{2} + \ldots + \binom{\frac{n-2}{2}}{\frac{n-2}{2}}\right)$$

For odd n, we have:

$$f(n) = (21-2n)\left(1 + \binom{n-3}{1} + \binom{n-4}{2} + \ldots + \binom{\frac{n-1}{2}}{\frac{n-3}{2}}\right)$$

We then calculate $f(n)$ for all possible values of n.

$$\begin{aligned}
f(0) &= 20 \\
f(2) &= 17(1) = 17 \\
f(3) &= 15(1) = 15 \\
f(4) &= 13(1+1) = 26 \\
f(5) &= 11(1+2) = 33 \\
f(6) &= 9(1+3+1) = 45 \\
f(7) &= 7(1+4+3) = 56 \\
f(8) &= 5(1+5+6+1) = 65 \\
f(9) &= 3(1+6+10+4) = 63 \\
f(10) &= 1(1+7+15+10+1) = 34
\end{aligned}$$

We sum all of these values to get the total number of subsets, which is $20 + 17 + 15 + 26 + 33 + 45 + 56 + 65 + 63 + 34 = \boxed{\mathbf{374}}$.

10. Let $\sqrt{64-x^2} - \sqrt{49-x^2} = 3$. What is the value of $\sqrt{64-x^2} + \sqrt{49-x^2}$?

Answer: $\boxed{\mathbf{5}}$

Solution:

Note that:

$$\begin{aligned}
\left(\sqrt{64-x^2} - \sqrt{49-x^2}\right)\left(\sqrt{64-x^2} + \sqrt{49-x^2}\right) &= (64-x^2) - (49-x^2) \\
&= 64 - 49 \\
&= 15
\end{aligned}$$

So, we have that:

$$\begin{aligned}\sqrt{64-x^2}+\sqrt{49-x^2} &= \frac{\left(\sqrt{64-x^2}-\sqrt{49-x^2}\right)\left(\sqrt{64-x^2}+\sqrt{49-x^2}\right)}{\sqrt{64-x^2}-\sqrt{49-x^2}} \\ &= \frac{15}{3} \\ &= \boxed{\mathbf{5}}\end{aligned}$$